$$s=\frac{(a+b)d'}{(a-b)(d'g-d)}\log\left[\frac{l\,t^{\sqrt{[2k(a-b)(d'g-d)]}/\sqrt{[(a+b)d']}}+l^{-t\sqrt{[2k(a-b)(d'g-d)]}/\sqrt{[(a+b)d']}}}{2}\right]$$

$$\text{or } s=\frac{(a+b)d'}{(a-b)(d'g-d)}\log(\cosh t\sqrt{[2k(a-b)(d'g-d)]}/\sqrt{[(a+b)d']}).$$

$$s=\frac{1}{p^2}\log(\cosh pt) \text{ where } p=\sqrt{\frac{2k(a-b)(d'g-d)}{(a+b)d'}}.$$

For four seconds, $s=\frac{1}{p^2}\log(\cosh 4p)$.

[See Bowser's *Analytic Mechanics*, page 314, ex. 5, where $v=0$ and $d=0$, of equation (3) above.]

Also solved by *ELMER SCHUYLER.*

AVERAGE AND PROBABILITY.

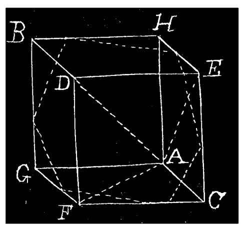
61. Proposed by COL. CLARKE.

A cube being cut at random by a plane, what is the chance that the section is a hexagon? [Erom Williamson's *Integeal Calculus.*]

Solution by LEWIS NEIKIRK, Graduate Student, University of Colorado, Boulder, Col.

I. PRELIMINARY INVESTIGATION.

Let S, the random section, be determined by the coördinates p, φ, and θ, θ being the angle between p and its projection on $ACFG$ and φ the angle between AC and the projection of p. Let P be the point of intersection of p and S. Also let p increase from zero for φ and $\theta<\frac{1}{2}\pi$. S, starting with three sides at A, gains three more, one at each of the corners C, G, and H; and loses three, one at each of the corners B, E, and F. φ and θ determine the order in which these gains and losses shall occur, and plainly S can be hexagonal only when the first loss is antedated by all three gains.



For p in the diagonal AD ($\varphi=\frac{1}{4}\pi$, $\theta=\cot^{-1}\sqrt{2}$), the three gains are simultaneous and are followed by three simultaneous losses. For p as an element of the area DAF ($\varphi=\frac{1}{4}\pi$, $\theta<\cot^{-1}\sqrt{2}$), one gain at H is followed by two more at C and G; then two losses at B and E, followed by one loss at F. For p as an element of the areas DAE and DAB ($\varphi<\frac{1}{4}\pi$ and $\theta=\tan^{-1}\cos\varphi$, and $\varphi>\frac{1}{4}\pi$ and $\theta=\tan^{-1}\sin\varphi$) there is a like sequence of gains and losses. So far it has been easy to enumerate the exact order in which all the gains and losses occur.

For p within the solid angle A—$DECF$ ($\varphi<\frac{1}{4}\pi$ and $\theta<\tan^{-1}\cos\varphi$) the first loss occurs at B, and the last gain at C; the order and place of the remaining gains and losses would be difficult to enumerate and is in any event immaterial

to the solution. A similar enumeration and statement to the angles A—$DBGF$ and A—$DBHE$. Evidently, then, our attention may be confined to any one of these angles, say A—$DECF$.

II. Solution.

As p increases within the solid angle A—$DECF$ ($\varphi<\frac{1}{4}\pi$ and $\theta<\tan^{-1}\cos\varphi$), S, starting at A with three sides, gains two sides successively at G and H, the order being immaterial; then for some values of φ and θ it will gain a third side at C before losing the first one at B; for other values this loss at B will andedate the third gain at C; in fact it may antedate the second of the two gains at G or H. For certain values of φ and θ between these extremes, this first loss and final gain will concur. In this last case p is an element of the area EAF ($\varphi<\frac{1}{4}\pi$ and $\theta=\tan^{-1}[\cos\varphi-\sin\varphi]$); for this area is plainly perpendicular to BC, and must therefore contain p when S reaches B and C simultaneously.

The number of cases which have the coördinates p, φ, and θ are $dpd\omega=\cos\theta dpd\theta d\varphi$, where ω is a solid angle with its vertex at A. The integration of p for the favorable cases extends from p_1 to p_2, where p_1 is the value of p when S reaches C, and p_2 is the value of p when S reaches B. The integration of θ extends from the plane EAF to the plane DAE, and the integration of φ from 0 to $\frac{1}{4}\pi$. The above limits may be calculated from the following spherical triangles in which the primed letters refer to points on a unit sphere, center at A, corresponding to points with unprimed letters in the figure.

It the right spherical triangle $P'C'F'$, $P'C'=\psi_1$, $F'C'=\varphi$, $P'F'=\theta$, and $\angle P'F'C'=90^\circ$. Then $p_1=a\cos\psi_1=a\cos\theta\cos\varphi$, where a is an edge of the cube.

In the spherical triangle $B'P'H'$, $B'F'=45^\circ$, $B'P'=\psi_2$, $P'F'=90^\circ-\theta$, and $\angle B'H'P'=90^\circ-\varphi$. Then $p_2=a\sqrt{2}\cos\psi_2$, $\cos\psi_2=(1/\sqrt{2})(\sin\theta+\cos\theta\sin\varphi)$.

$\therefore\ p_2=a(\sin\theta+\cos\theta\sin\varphi)$.

In the right spherical triangle $F'P'C'$ right angled at C', $P'C'=\theta_1$, $F'C'=45^\circ-\varphi$. and $\angle P'F'C'=\cot^{-1}(1/\sqrt{2})$.

Then $\tan\theta_1=\sqrt{2}\sin(45^\circ-\varphi)=\cos\varphi-\sin\varphi$. Then $\theta_1=\tan^{-1}(\cos\varphi-\sin\varphi)$.

In the right spherical triangle $P'H'E'$ right angled at E', $H'E'=45^\circ$, $P'H'=90^\circ-\theta_2$, and $\angle P'H'E'=\varphi$. Then $\tan\theta_2=\cos\varphi$, $\theta_2=\tan^{-1}\cos\varphi$.

All the favorable cases

$$F=12\int_0^{\frac{1}{4}\pi}\int_{\theta_1}^{\theta_2}\int_{p_1}^{p_2}\cos\theta dpd\theta d\varphi=12a(\sqrt{3}\tan^{-1}\tfrac{1}{3}\sqrt{3}-\sqrt{2}\tan^{-1}\tfrac{1}{2}\sqrt{2}).$$

The integration for the total number of cases extends for p from 0 to p', where p' is the value of p when S reaches D; for θ, from 0 to $\frac{1}{2}\pi$, and for φ from 0 to $\frac{1}{2}\pi$.

In the spherical triangles $D'AP'$ and $D'AF'$, $D'A=\psi'$, $P'A=\theta$, $D'P'=\beta$, $\angle D'AP'=\alpha$, $F'A=(45^\circ-\varphi)$, $F'D'=\tan^{-1}(1/\sqrt{2})$, $\angle D'F'A=90$, and $\angle D'AF'=(90^\circ-\alpha)$.

Then $\cos\psi'=\sqrt{\tfrac{2}{3}}\cos(45^\circ-\varphi)=(1/\sqrt{3})(\cos\varphi+\sin\varphi)$, $\tan\alpha=\sqrt{2}\sin(45-\varphi)$
$=\cos\varphi-\sin\varphi$,

$p = a\sqrt{3}\cos\beta = a\sqrt{3}\,(\cos\psi'\cos\theta + \sin\psi'\sin\theta\cos\alpha)$
$= a[\cos\theta(\cos\varphi + \sin\varphi) + \sin\theta]$.

The total number of cases is

$$T = 4\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{p'} \cos\theta\, dp\, d\theta\, d\varphi = 3\pi a.$$

Therefore the probability is $P = E/T = 4/\pi(\sqrt{3}\tan^{-1}\tfrac{1}{3}\sqrt{3} - \sqrt{2}\tan^{-1}\tfrac{1}{2}\sqrt{2})$.

NOTE.—I wish to acknowledge my indebtedness to Professor DeLong and Mr. Frank Giffin for looking over this solution and making valuable suggestions.

MISCELLANEOUS.

70. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Prove $\tan^{-1}x = \frac{1}{2i}\left(\log\frac{x-i}{x+i}\right)$, and thence that $\pi = (2/i)\log(i)$.

I. Solution by GUY B. COLLIER and HAROLD C. FISKE, Class 1901, Union College, Schenectady, N. Y., and the PROPOSER.

Consider the integral,

$$\int\frac{dx}{1+x^2} = \tan^{-1}x \ldots\ldots(1).$$

Integrate the left member by partial fractions

$$\int\frac{dx}{1+x^2} = \frac{1}{2i}\int\frac{dx}{x-i} - \frac{1}{2i}\int\frac{dx}{x+i} = \frac{1}{2i}\log\frac{x-i}{x+i} \ldots\ldots(2).$$

$\therefore$ From (1) and (2),

$$\tan^{-1}x = \frac{1}{2i}\log\frac{x-i}{x+i}.$$

When $x=1$ this becomes

$$\tfrac{1}{4}\pi = \frac{1}{2i}\log\frac{1-i}{1+i} = \frac{1}{2i}\log(-i).$$

$\pi = (2/i)\log(-i)$, it should have been.

II. Solution by R. E. GAINES, A. M., Professor of Mathematics, Richmond College, Richmond, Va.

We have the identity

$$\frac{1}{1+x^2} = \frac{1}{2i}\left(-\frac{1}{i-x} - \frac{1}{i+x}\right).\ \therefore \int\frac{dx}{1+x^2} = \frac{1}{2i}\log\left(\frac{i-x}{i+x}\right) + c.$$